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On conformal Killing 2-form of the electromagnetic field

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Abstract

Let $D : C^\infty \Lambda^p M \to C^\infty(T^*M \otimes \Lambda^p M)$ be the first order linear differential operator on an n-dimensional ($1 \leq p \leq n-1$) pseudo-Riemannian manifold (M, g). We have by the representation theory of orthogonal group, that the tangent bundle of this operation decomposes into the orthogonal and irreducible sum of forms of degree $p+1$ (which gives the exterior differential d), the forms of degree $p-1$ (defining the codifferential d*) and the trace-free part of the partial symmetrization (the corresponding first order operator is denoted by D). The general forms in the kernel of D are closely related to conformal Killing vector fields, called conformal Killing p-forms, while those in kernel of d are called closed conformal Killing p-forms or, according to another terminology, planar p-forms. In particular an arbitrary planar 1-form ω is dual (by g) to the special concircular vector field ξ. We consider some local properties for the closed conformal Killing p-forms. As an application we present examples of decomposition into irreducible components for the electromagnetic field 2-form ω and its covariant derivative in four-dimensional space–time. In particular, we prove that the energy–momentum tensor T of the electromagnetic field is a symmetric conformal Killing tensor if the electromagnetic field 2-form ω is a conformal Killing form.

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1. Introduction

In this article we consider an n-dimensional C^∞-manifold M and endow it with a pseudo-Riemannian metric g. A pair (M, g) is called a *pseudo-Riemannian manifold.*

Denote by ∇ the Levi–Civita connection and $\mathcal{L}$ the Lie operator on M. A vector field ξ on M is said to be *conformal Killing vector field* if it satisfies the following equation:

$$(\mathcal{L}_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 2\rho g(X, Y)$$

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for any $X, Y \in C^\infty TM$ and a smooth function $\rho : M \to \mathbf{R}$. In particular, if the function ρ vanishes identically, then the vector field ξ satisfies the following Killing's equation: $\mathcal{L}_\xi g = 0$. In this case the vector field ξ is called a *Killing vector field.*

Killing differential forms are the generalization of Killing vector fields, and also this generalization is carried out in two different directions.

On our fixed manifold M, we denote by S^pM the bundle of symmetric bilinear differential p-forms on M. Hence $C^\infty S^pM$, the space of all C^∞-sections of S^pM, is the space of symmetric bilinear differential forms of degree p.

As it is well known (see [5,8]), a symmetric bilinear differential form $\varphi \in C^\infty S^pM$ is said to be *Killing symmetric form*, or according to another terminology, to be *Killing tensor field* if it satisfies $\delta^*\varphi = 0$ for the symmetric differentiation operator $\delta^* : C^\infty S^pM \to C^\infty S^{p+1}M$ (see [1]). It is obvious that this generalizes the usual definition for $p = 1$.

The interest in such Killing symmetric differential forms appeared, from the point of view of relativistic physics, due to the fact that they are connected with pth-order first integrals of the differential equations of geodesics on a Reimannian or pseudo-Riemannian manifold (M, g), homogeneous polynomial functions and with the separation of the variables in partial differential equations, e.g. in Hamilton–Jacobi's equations (see [5,8]).

Side by side with Killing symmetric differential forms on a pseudo-Riemannian manifold (M, g), they examine conformal Killing symmetric differential forms (see [8]). Now we can denote by S_0^2M the vector bundle of trace-free symmetric bilinear differential 2-forms on M. A symmetric form $\varphi' \in C^\infty S_0^2M$ is said to be *conformal Killing symmetric form* or, according to another terminology, to be *conformal Killing tensor field* if it satisfies $\delta^*\varphi' = -S^3(g \otimes \varphi')$ where $S^3(g \otimes \varphi')$ is the completely symmetric part of $g \otimes \varphi'$.

As it is well known, if there is a conformal Killing symmetric differential form on a pseudo-Riemannian manifold (M, g), then the differential equations of isotropic geodesics on (M, g) have a quadratic first integral.

At present large material on the geometry of Killing symmetric differential forms and of space–time, carrying the fields of this kind, has been amassed. So the structure of Killing symmetric differential form is known on a flat space (see [18]).

On the other hand, we denote by $\Lambda^pM (p = 1, ..., n)$ the pth exterior power $\Lambda^p(T^*M)$ of the contagent bundle T^*M of M. Hence $C^\infty\Lambda^pM$, the space of all C^∞-sections of Λ^pM, is the space of exterior differential forms of degree p.

As it is well known (see [4,8]), an exterior differential form $\omega \in C^\infty\Lambda^pM$ is said to be *Killing differential p-form*, or according to another terminology, to be *Killing–Yano tensor field* if it satisfies $d\omega = (p+1)\nabla\omega$ for the exterior differential operator $d : C^\infty C^\infty\Lambda^pM \to C^\infty\Lambda^{p+1}M$.

Let us state that the differential equations of geodesics have a quadratic first integral on a pseudo-Riemannian manifold (M, g) if there is a nontrivial Killing–Yano tensor field (see [5,8]).

Side by side with conformal Killing symmetric bilinear differential forms, they examine conformal Killing exterior differential p-forms for $1 \le p \le n$. The concept of a conformal Killing exterior differential p-form was introduced by Kashiwada and Tachibana (see [7,16]). They generalized some results of a conformal Killing vector field to a conformal

Killing exterior differential p-form. An exterior differential form $\omega \in C^{\infty}\Lambda M$ is said to be *conformal Killing*, if there is an exterior differential $(p-1)$-form θ called the associated form such that

$$\begin{aligned}&(\nabla_Y\omega)(X, X_2, \dots, X_p) + (\nabla_X\omega)(Y, X_2, \dots, X_p)\\&= 2g(Y, X)\theta(X_2, \dots, X_p) - \sum_{a=2}^{p}(-1)^a[g(Y, X_a)\theta(X, X_2, \dots, \hat{X}_a, \dots, X_p)\\&+g(X, X_a)\theta(Y, X_2, \dots, \hat{X}_a, \dots, X_p)]\end{aligned}$$

for any $Y, X, X_1, \dots, X_a, \dots, X_p \in C^{\infty}TM$, where $\hat{X}_a$ means that X_a is omitted. It is obvious that this generalizes the usual definition for $p = 1$.

Large material on the geometry of Killing and of conformal Killing exterior differential p-forms has been amassed.

In Section 2 we base the study of the seven subspaces of the vector space $\mathbf{\Omega}^p(M, \mathbf{R})$ of exterior differential p-forms on (M, g) from the stand-point of first order linear differential operators from $C^{\infty}\Lambda^pM$ to $C^{\infty}(T^*M \otimes \Lambda^pM)$ and of representations of the orthogonal group $O(g_x)$ for any $x \in M$. In Section 3 we consider local properties of closed conformal Killing p-forms. In Section 4 we present examples of decomposition into irreducible components for the electromagnetic field 2-form ω and its covariant derivative. In particular we prove that the energy–momentum tensor T of the electromagnetic field is a symmetric conformal Killing tensor if the electromagnetic field 2-form ω is a conformal Killing form.

Some results of this work were announced in [11,12,14]. We shall omit any proofs that are simple computation, or that can be found in the literature.

2. The classification of differential forms from the point of view of differential opertors and group representations

Consider a pseudo-Riemannian manifold (M, g). At every point $x \in M$ metric g defines a square form $q = g_x$ of some signature on the tangent space $E = T_xM$. Denote by $O(q)$ the group of linear transformations $\mathcal{A}$ under which the quadratic form q is invariant; it means the following: $q(\mathcal{A}, \mathcal{A}) = q$. It is customary to call $O(q)$ an orthogonal group. If $\mathcal{A}$ belongs to the orthogonal group $O(q)$ determined by the form q, then ${}^{\mathrm{tr}}\mathcal{A}^{-1} = \mathcal{A}$ and, hence, the $O(q)$-modules E and E^* are isomorphic. Therefore, we consider only the tensor powers of the space E and assume, in addition, that $E^{(k)} = \otimes^k E$. The quadratic form q induces a scalar product on $E^{(k)}$ defined by the formula

$$\langle v_1 \otimes \cdots \otimes v_k, w_1 \otimes \cdots \otimes w_k\rangle = \prod_{i=1}^{k} q(v_i, w_i) \tag{2.1}$$

for arbitrary $v_1, \dots, v_k, w_1, \dots, w_k \in E$. The orthogonal group $O(q)$ also acts in $E^{(k)}$ according to the rule

$$\mathcal{A}(v_1 \otimes \cdots \otimes v_k) = \mathcal{A}(v_1) \otimes \cdots \otimes \mathcal{A}(v_k),$$

which permits $E^{(k)}$ to be regarded as a representation space of the orthogonal group $O(q)$.

Recall that a representation of the group G of linear transformation in a vector space V is said to be irreducible if each subspace L in V that is invariant with respect to all linear transformations in the group coincides with either the zero subspace or the whole of V.

To find all of the subspaces in $E^{(k)}$ that are irreducible with respect to the action of the group $O(q)$, apply the following two Weyl theorems on invariants.

Theorem A. *Nonzero linear forms on $E^{(k)}$, invariant with respect to $O(q)$, exist only for even values of k. In this case, they are generated by the elementary forms below:*

$$\Phi_{\sigma,\tau} : v_1 \otimes \cdots \otimes v_{2r} \to \prod_{i=1}^{k} q(v_{\sigma(i)}, v_{\tau(i)}),$$

where σ and τ are injective mappings of the set $\{1, \dots, r\}$ into $\{1, \dots, 2r\}$ with nonintersecting images.

Theorem B. *Let L be a vector subspace in the tensor algebra $T(E)$ of the space E such that the space of $O(q)$-invariant quadratic forms on L is one-dimensional. Then L is irreducible with respect to $O(q)$.*

Finally, consider an $O(q)$-equivariant mapping $\mathrm{tr}_{ij} : E^{(k)} \to E^{(k-2)}$ for $k \geq j > i \geq 1$, defined by the formula:

$$\begin{aligned}&\mathrm{tr}_{ij}(v_1^* \otimes \cdots \otimes v_i^* \otimes \cdots \otimes v_j^* \otimes \cdots \otimes v_k^*)\\ &\quad = q(v_i^*, v_j^*)(v_1^* \otimes \cdots \otimes \hat{v}_i^* \otimes \cdots \otimes \hat{v}_j^* \otimes \cdots \otimes v_k^*).\end{aligned}$$

Then every tensor field $K \in C^\infty T^{(p,0)}M$ can be represented by the orthogonal sum (see [19])

$$\begin{aligned}K(X_1, \dots, X_p) = &\sum_{i<j} g(X_i, X_j) K_{ij}(X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_p)\\ &+ K_0(X_1, \dots, X_p)\end{aligned}$$

for any vector fields $X_1, \dots, X_p \in C^\infty TM$ and $\mathrm{tr}_{ij} K_0 = 0$ for all $i < j$ and $i, j = 1, \dots, p$.

We now consider the space $\mathrm{Diff}_1(\Lambda^p M, T^*M \otimes \Lambda^p M)$ of first order linear differential operators from $C^\infty \Lambda^p M$ to $C^\infty(T^*M \otimes \Lambda^p M)$.

Let $\mathcal{W}$ be a subspace of the tensor space $T^*M \otimes \Lambda^p M$. We suppose that the subspace $\mathcal{W}$ is point-wise irreducible under the action of $O(q)$ (see [19]). Further we shall call a differential operator $\mathrm{D} \in \mathrm{Diff}_1(\Lambda^p M, T^*M, \otimes \Lambda^p M)$ *fundamental*, if its main symbol is a projector to the $\mathcal{W}$ (see [10,1]).

Having used the representation theory of orthogonal group (see [19]), we shall search all fundamental differential operators $\mathrm{D} : C^\infty \Lambda^p M \to C^\infty(T^*M \otimes \Lambda^p M)$ in case when (M, g) is an n-dimensional pseudo-Riemannian manifold.

Now we shall prove the following

Theorem 2.1. *Consider an n-dimensional pseudo-Riemannian manifold (M, g) and the space* $\mathrm{Diff}_1(\Lambda^p M, T^*M\otimes\Lambda^p M)$ *of first order linear differential operators from* $C^\infty\Lambda^p M$ *to* $C^\infty(T^*M\otimes\Lambda^p M)$. *There are only three fundamental differential operators*:

$$\mathrm{D}_1=\frac{1}{p+1}\mathrm{d},\quad \mathrm{D}_2=\frac{1}{m-p+1}g\,\Box\,\mathrm{d}^*,\quad \mathrm{D}_3=\nabla-\frac{1}{p+1}\mathrm{d}-\frac{1}{m-p+1}g\,\Box\,\mathrm{d}^*,$$

where ∇, d, d^* *mean the operators of covariant differential, exterior differential, exterior co-differential respectively, and*

$$(g\,\Box\,\mathrm{d}^*\omega)(X_0, X_1,\dots,X_p)=\sum_{a=1,\dots,p}(-1)^a g(X_0,X_a)(\mathrm{d}^*\omega)(X_1,\dots,\hat{X}_a,\dots,X_p)$$

for an arbitrary differential p-form $\omega\in C^\infty\Lambda^p M$ *and any vector fields* $X_0, X_1,\dots,X_p\in C^\infty TM$. *The kernel of* D_1 *consists of closed p-forms, the kernel of* D_2 *consists of co-closed p-forms and the kernel of* D_3 *consists of conformal Killing p-forms, making up three subspaces* $\mathbf{D}^p(M,\mathbf{R})$, $\mathbf{F}^p(M,\mathbf{R})$ *and* $\mathbf{T}^p(M,\mathbf{R})$ *of the vector space* $\mathbf{\Omega}^p(M,\mathbf{R})$ *of exterior differential p-forms on* (M, g).

Proof. At first we shall demonstrate that the induced representation of $O(q)$ on $T^*M\otimes\Lambda^p M$ (for $1\le p\le m-1$) has three irreducible components

$$T^*M\otimes\Lambda^p M=\Lambda^{p+1}M\oplus(\mathbf{R}g\Box\Lambda^{p-1}M)\oplus(\ker\mathrm{tr}_{12}\cap\ker\Lambda^{p+1}).$$

On the one hand, the tensor space $T^*M\otimes\Lambda^p M$ is the sum of two point-wisely orthogonal components $T^*M\otimes\Lambda^p M=\Lambda^{p+1}M\oplus\ker\Lambda^{p+1}$. □

On the other hand, the tensor space $\ker\Lambda^{p+1}$ can be expanded into an orthogonal sum of two point-wisely $O(q)$-invariant subspace

$$\ker\Lambda^{p+1}=(\mathbf{R}g\Box\Lambda^{p-1}M)\oplus(\ker\mathrm{tr}_{12}\cap\ker\Lambda^{p+1}).$$

Here, with respect to the definition, we suppose, that

$$(g\,\Box\,\omega)(X_0, X_1,\dots,X_p)=\sum_{a=1}^{p}(-1)^a g(X_0,X_a)\theta(X_1,\dots,\hat{X}_a,\dots,X_p),$$

where $\theta\in C^\infty\Lambda^{p-1}M$ and $X_0,\dots,X_p\in C^\infty TM$. Then the following decomposition in orthogonal complements holds:

$$T^*M\otimes\Lambda^p M=\Lambda^{p+1}\oplus(\mathbf{R}g\Box\Lambda^{p-1}M)\oplus(\ker\mathrm{tr}_{12}\cap\ker\Lambda^{p+1}).\tag{2.2}$$

This orthogonal sum is preserved under the representation of $O(q)$ on $T^*M\otimes\Lambda^p M$. The induced representation of $O(q)$ on $\Lambda^{p+1}M$, $\mathbf{R}g\Box\Lambda^{p-1}M$ and $\mathrm{kr}\,\mathrm{tr}_{12}\cap\ker\Lambda^{p+1}$ are point-wisely irreducible. The number of point-wisely irreducible components of representation of $O(q)$ on $T^*M\otimes\Lambda^p M$ is equal to the dimensional of the space of quadratic forms on the space $E^*\otimes\Lambda^p E$ (for $1\le p\le m-1$). We can now show that this dimensional is equal to 3. Define Q_1, Q_2 and Q_3 by

$$Q_1(K)=g(K,K),\qquad Q_2(K)=g(\mathrm{tr}_{12}K,\ \mathrm{tr}_{12}K),$$

$$Q_3 = \sum_{i_1,\dots,i_{p+1}=1}^{n} K(e_{i_1}, e_{i_2}, e_{i_3}, \dots, e_{i_{p+1}}) K(e_{i_2}, e_{i_1}, e_{i_3}, \dots, e_{i_{p+1}})$$

for $K \in C^\infty(T^*M \otimes \Lambda^p M)$. It is clear that Q_1, Q_2 and Q_3 are quadratic invariants of the representation of $O(q)$ on $T^*M \otimes \Lambda^p M$. They are linearly independent, it can be proved directly. Thus the representation of $O(q)$ on $T^*M \otimes \Lambda^p M$ has precisely three point-wisely irreducible components, namely $\Lambda^{p+1}M$, $\mathbf{R}g\Box\Lambda^{p-1}M$ and $\ker \mathrm{tr}_{12} \cap \ker \Lambda^{p+1}$.

From (2.2) a familiar point-wisely irreducible under the action of $O(q)$ on $\otimes^2 T^*M$ can be deduced (see [1]): $\otimes^2 T^*M = \Lambda^2 M \oplus S_0^2 M \oplus \mathbf{R}g$.

Orthogonal projections on irreducible components in the expansion (2.2) are defined by the formula

$$\mathrm{Id}_{T^*M\otimes\Lambda^p M} = \mathrm{Pr}_{\Lambda^{p+1}M} + \mathrm{Pr}_{\mathbf{R}g\Box\Lambda^{p-1}M} + \mathrm{Pr}_{\ker \mathrm{tr}_{12}\cap\ker\Lambda^{p+1}},$$

where we have for an arbitrary field $K \in C^\infty(T^*M \otimes \Lambda^p M)$

$$\begin{aligned}[\mathrm{Pr}_{\Lambda^{p+1}M} K](X_0, X_1, \dots, X_p) = \frac{1}{p+1}[&K(X_0, X_1, \dots, X_{p-1}, X_p) \\ &+K(X_1, \dots, X_{p-1}, X_p, X_0) \\ &+\cdots + K(X_p, X_0, \dots, X_{p-1})],\end{aligned}$$

$$\begin{aligned}&[\mathrm{Pr}_{\mathbf{R}g\Box\Lambda^{p-1}M} K](X_0, X_1, \dots, X_p) \\ &= \frac{1}{m-p+1}\left[\sum_{a=1}^{p} (-1)^a g(X_0, X_a)(\mathrm{tr}_{12}K)(X_1, \dots, \hat{X}_a, \dots, X_p)\right],\end{aligned}$$

$$\begin{aligned}&[\mathrm{Pr}_{\ker \mathrm{tr}_{12}\cap\ker\Lambda^{p+1}} K](X_0, X_1, \dots, X_{p-1}, X_p) \\ &= K(X_0, X_1, \dots, X_{p-1}, X_p) - \frac{1}{p+1}[K(X_0, X_1, \dots, X_{p-1}, X_p) \\ &\quad +K(X_1, \dots, X_{p-1}, X_p, X_0) + \cdots + K(X_p, X_0, X_1, \dots, X_{p-1}] \\ &\quad -\frac{1}{m-p+1}\left[\sum_{a=1}^{p} (-1)^a g(X_0, X_a)(\mathrm{tr}_{12}K)(X_1, \dots, \hat{X}_a, \dots, X_p)\right].\end{aligned}$$

It is easy to perceive that $\mathrm{D}_1 = (p+1)^{-1}\mathrm{d}$ is the first fundamental operator from $C^\infty \Lambda^p M$ to $C^\infty(T^*M \otimes \Lambda^p M)$.

Let us introduce into inspection a differential operator

$$\mathrm{D}_2 : C^\infty \Lambda^p M \to C^\infty(T^*M \otimes \Lambda^p M),$$

which is a composition of two operators: the exterior co-differential

$$\mathrm{d}^* : C^\infty \Lambda^p M \to C^\infty \Lambda^{p-1} M$$

and the algebraic operator

$$\theta \in C^\infty \Lambda^{p-1} M \to \frac{1}{m-p+1} g\Box\theta \in C^\infty(T^*M \otimes \Lambda^p M).$$

Let us remind that the value of the main symbol $\sigma(\mathrm{d}^*)$ of the exterior co-differential d^* on an arbitrary 1-form $\nu \in C^\infty T^*M$ is the following linear mapping:

$$\sigma(\mathrm{D}^*)\nu : \omega \in C^\infty \Lambda^p M \to -i_\nu\omega \in C^\infty \Lambda^{p+1} M,$$

where i_ν is an interior product. Then the value of the main symbol of the operator D_2 on $\nu \in C^\infty T^*M$ will be according to (2.5), the following mapping:

$$\sigma(\mathrm{D}_2)\nu : \omega \in C^\infty \Lambda^p M \to -\frac{1}{m-p+1} g\Box(i_\nu\omega) \in C^\infty \Lambda^{p+1} M.$$

Thus, the main symbol $\sigma(\mathrm{D}_2)$ of the operator D_2 will be the projection on $O(q)$ — irreducible component $\mathbf{R}g\Box\Lambda^{p-1}M$ of the decomposition (2.2). Therefore the operator

$$\mathrm{D}_2 = \frac{1}{m-p+1} g \,\Box\, \mathrm{d}^*$$

will be the second fundamental operator from $C^\infty \Lambda^p M$ to $C^\infty(T^*M \otimes \Lambda^p M)$.

In conclusion we can consider the following differential operator of the first degree:

$$\mathrm{D}_3 = \nabla - \frac{1}{p+1}\mathrm{d} - \frac{1}{m-p+1} g \,\Box\, \mathrm{d}^*.$$

The value of $\nu \in C^\infty T^*M$ of its main symbol $\sigma(\mathrm{D}_3)$ is the following linear mapping:

$$\sigma(\mathrm{D}_3)\theta : \omega \in C^\infty \Lambda^p M$$
$$\to \left(\nu \otimes \omega - \frac{1}{p+1}\nu \wedge \omega + \frac{1}{m-p+1} g\Box(i_\nu\omega)\right) \in C^\infty(T^*M \otimes \Lambda^p M).$$

It is easy to perceive that the main symbol $\sigma(\mathrm{D}_3)$ of the operator D_3 will be the projection on the irreducible expansion component (2.2). Therefore D_3 will be the third fundamental operator from $C^\infty \Lambda^p M$ to $C^\infty(T^*M \otimes \Lambda^p M)$.

Thus we have demonstrated, that there are only three fundamental operators of order one determined on $C^\infty \Lambda^p M$. It will enable to dedicate seven different subspaces of the space $\mathbf{\Omega}^p(M, \mathbf{R})$ of exterior differential forms of degree p on (M, g).

A p-form $\omega (p \geq 1)$ is said to be closed if it satisfies $\mathrm{d}\omega = 0$ that is equivalent to the following condition $\omega \in \ker \mathrm{D}_1$. The set of closed p-forms is a subspace of the vector space $\mathbf{\Omega}^p(M, \mathbf{R})$. We shall denote it by $\mathbf{D}^p(M, \mathbf{R})$.

It is not easy to conclude that the kernel of the second fundamental differential operator $\mathrm{D}_2 = (n-p+1)^{-1} g \,\Box\, \mathrm{d}^*$ consists of the coclosed p-forms and the set of such forms is a subspace of the space $\mathbf{\Omega}^p(M, \mathbf{R})$. We can denote it by $\mathbf{F}^p(M, \mathbf{R})$.

Let us consider the kernel of the third fundamental differential operator D_3. We suppose that $p = 1$, then for any $\omega \in C^\infty T^*M$ we shall have

$$\begin{aligned}(\mathrm{D}_3\omega)(X, Y) &= \frac{1}{2}[(\nabla_X\omega)Y + (\nabla_Y\omega)X] + \frac{1}{n} g(X, Y)\,\mathrm{d}^*\omega \\ &= \frac{1}{2}(\mathcal{L}_\xi g)(X, Y) - \frac{1}{n}(\operatorname{div}\xi) g(X, Y), \qquad (2.3)\end{aligned}$$

where ξ is the vector field dual (by g) to the 1-form ω. Consequently, the kernel of the operator D_3 for $p = 1$ consists of conformal Killing vector fields, or according to another terminology, of *infinitesimal conformal transformations* in (M, g) (see [2]).

We shall return to a general case when $p \geq 1$. It is obvious that the kernel of the operator D_3 consists of such p-forms $\omega \in C^\infty \Lambda^p M$, that

$$\nabla\omega = \frac{1}{p+1}\mathrm{d}\omega + \frac{1}{m-p+1} g \,\square\, \mathrm{d}^*\omega. \tag{2.4}$$

Then by analogy with (2.3) we shall find the expression for the following sum of covariant derivatives:

$$\begin{aligned}
&(\nabla_{X_0}\omega)(X_1, X_2, \dots, X_p) + (\nabla_{X_1}\omega)(X_0, X_2, \dots, X_p) \\
&= \frac{1}{p+1}[(\mathrm{d}\omega)(X_0, X_1, X_2, \dots, X_p) + (\mathrm{d}\omega)(X_1, X_0, X_2, \dots, X_p)] \\
&\quad + \frac{1}{m-p+1}\Bigg[- g(X_0, X_1)(\mathrm{d}^*\omega)(X_2, \dots, X_p) \\
&\quad + \sum_{a=2}^{p}(-1)^a g(X_0, X_i)(\mathrm{d}^*\omega)(X_1, X_2, \dots, \hat{X}_a, \dots, X_p) \\
&\quad - g(X_1, X_0)(\mathrm{d}^*\omega)(X_2, \dots, X_p) \\
&\quad + \sum_{a=2}^{p}(-1)^a g(X_1, X_a)(\mathrm{d}^*\omega)(X_0, X_2, \dots, \hat{X}_a, \dots, X_p)\Bigg] \\
&= -\frac{2}{m-p+1} g(X_0, X_1)(\mathrm{d}^*\omega)(X_2, \dots, X_p) \\
&\quad + \frac{1}{m-p+1}\sum_{a=2}^{p}(-1)^a [g(X_0, X_a)(\mathrm{d}^*\omega)(X_1, X_2, \dots, \hat{X}_a, \dots, X_p) \\
&\quad + g(X_1, X_a)(\mathrm{d}^*\omega)(X_0, X_2, \dots, \hat{X}_a, \dots, X_p)].
\end{aligned}$$

Taking the designations

$$\theta = -\frac{1}{m-p+1}\mathrm{d}^*\omega, \tag{2.5}$$

we come to the following equation:

$$\begin{aligned}
&(\nabla_{X_0}\omega)(X_1, X_2, \dots, X_p) + (\nabla_{X_1}\omega)(X_0, X_2, \dots, X_p) \\
&= 2g(X_0, X_1)\theta(X_2, \dots, X_p) - \sum_{a=2}^{p}(-1)^a [g(X_0, X_a)\theta(X_1, X_2, \dots, \hat{X}_a, \dots, X_p) \\
&\quad + g(X_1, X_a)\theta(X_0, X_2, \dots, \hat{X}_a, \dots, X_p)].
\end{aligned} \tag{2.6}$$

Eq. (2.6) determines the p-form ω as a conformal Killing p-form on a Riemannian manifold (M, g). This definition does not depend on the signature of the metric g. These facts imply that any p-form ω on a pseudo-Riemannian manifold (M, g), belonging to the kernel of the fundamental operator D_3, is a conformal Killing p-form. By Eq. (2.4) one can prove that the set of all conformal Killing forms of degree p on (M, g) is a subspace of the vector space $\mathbf{\Omega}^p(M, \mathbf{R})$. We shall denote it as $\mathbf{T}^p(M, \mathbf{R})$; the theorem is proved.

Remark. *Let (M, g) be a Riemannian manifold. Natural differential operators of order one determined on $C^\infty \Lambda^p M$ are considered by Bourguignon (see* [2]). *He had proved the existence of three basis natural differential operators, but only two of them were recognized. Main symbols of these operators are projectors to point-wisely irreducible subspaces of $T^*M \otimes \Lambda^p M$. According to Theorem* 2.1 *we conclude that $\{\mathrm{D}_1, \mathrm{D}_2, \mathrm{D}_3\}$ is the basis of natural differential operators of order one determined on $C^\infty \Lambda^p M$.*

Furthermore we can classify the differential p-forms from the point of view of fundamental differential operators.

One the one hand, if a conformal Killing p-form ω belongs to $\ker \mathrm{D}_2$, it is said to be Killing and characterized by the equation $\mathrm{D}_3\omega = \nabla\omega$. The set of Killing p-forms on (M, g) is a subspace of the vector space $\mathbf{\Omega}^p(M, \mathbf{R})$. We shall denote it as $\mathbf{K}^p(M, \mathbf{R})$.

On the other hand, if a conformal Killing p-form ω belongs to $\ker \mathrm{D}_1$, then it is called a closed conformal Killing p-form, or according to another terminology planar p-form (see [11]). We can denote the vector space of closed conformal Killing p-forms as $\mathbf{P}^p(M, \mathbf{R})$. For a closed conformal Killing 1-form ω from (2.4) we obtain that $\nabla\omega = -n^{-1}(\mathrm{d}^*\omega)g$. Therefore our 1-form ω is dual (by g) to the special concircular vector field ξ defined by the condition $\nabla\xi = -n^{-1}(\mathrm{d}^*\xi)\,\mathrm{id}$ (see [20]).

It is well-known (see [4]) that $\omega \in \ker \mathrm{d} \cap \ker \mathrm{d}^*$ is a necessary and sufficient condition for a p-form ω to be harmonic on a Riemannian manifold (M, g). Hence a harmonic p-form ω belongs to $\ker \mathrm{D}_1 \cap \ker \mathrm{D}_2$ on a Riemannian manifold (M, g). The vector space $\mathbf{H}^p(M, \mathbf{R})$ of the harmonic p-forms as well known as the vector space $\mathbf{K}^p(M, \mathbf{R})$ of the Killing forms of order p on a Riemannian manifold (M, g). We also denote the $\mathbf{R}$-module of "harmonic forms" of degree p on a pseudo-Riemannian manifold by $\mathbf{H}^p(M, \mathbf{R})$.

In conclusion, we show that the following inclusions exist between the various subspaces:

$$\begin{array}{ccccc}
\mathbf{P}^p(\mathrm{M},\mathbf{R}) & \subset & \mathbf{D}^p(\mathrm{M},\mathbf{R}) & \supset & \mathbf{H}^p(\mathrm{M},\mathbf{R}) \\
\cap & & \cap & & \cap \\
\mathbf{T}^p(\mathrm{M},\mathbf{R}) & \subset & \Omega^p(\mathrm{M},\mathbf{R}) & \supset & \mathbf{F}^p(\mathrm{M},\mathbf{R}) \\
 & \supset & & \subset & \\
 & & \mathbf{K}^p(\mathrm{M},\mathbf{R}) & &
\end{array}$$

3. Local geometry of closed conformal Killing forms

We denote by U an arbitrary neighborhood with local coordinate system $\{x^1, \ldots, x^n\}$ on a pseudo-Riemannian manifold (M, g) and use the summation convention. Let $\{\partial/\partial x^1, \ldots, \partial/\partial x^n\}$ be the field of the natural frame and $\{\mathrm{d}x^1, \ldots, \mathrm{d}x^n\}$ the field of the natural co-frame. We represent tensors by their components with respect to the natural frame or co-frame and use the summation convention. For example, the components of the metric tensor g have the form $g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j)$, and a p-form $\omega \in C^\infty \Lambda^p M$ is written as

$$\omega = \frac{1}{p!}\omega_{i_1 i_2 \cdots i_p}\,\mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_p},$$

where latin indices run over the range $\{1, 2, \ldots, n\}$ and '$\wedge$' means exterior multiplication.

Define a symmetric tensor field φ, of type (0,2), by the following equation:

$$\varphi_{ij} = g^{l_2 k_2} \cdots g^{l_p k_p}\omega_{i l_2 \cdots l_p}\ \omega_{j k_2 \cdots k_p} = \omega_{i l_2 \cdots l_p}\omega_j^{\ l_2 \cdots l_p},$$

where g^{jk} are the components of the metric tensor g.

Suppose ω is a conformal Killing p-form, then we have (2.4). Applying ∇ to φ_{ij} and making use of (2.4), we have

$$\nabla_k \varphi_{ij} = \nabla_{[k}\omega_{i l_2 \cdots l_p]}\omega_j^{\ l_2 \cdots l_p} + \omega_i^{\ l_2 \cdots l_p}\nabla_{[k}\omega_{j l_2 \cdots l_p]} + \frac{1}{m-p+1}\{g_{kj}\Theta_i + g_{ki}\Theta_j - (p-1)(\Theta_{ikj} + \Theta_{jki})\},$$

where

$$\Theta_1 = g^{l_2 k_2} \cdots g^{l_p k_p}\omega_{l l_2 \cdots l_p}\nabla_k \omega^k_{\ k_2 \cdots k_p}, \qquad \Theta_{jlk} = g^{l_3 k_3} \cdots g^{l_p k_p}\omega_{j l l_2 \cdots l_p}\nabla_j \omega^j_{\ k k_3 \cdots k_p}.$$

By a simple substitution one can verify the following equations:

$$(\nabla_X \varphi)(Y, Z) + (\nabla_Y \varphi)(Z, X) + (\nabla_Z \varphi)(X, Y) = -\frac{2}{m-p+1}\{g(X, Z)(\delta\varphi)(Y) + g(Y, Z)(\delta\varphi)(X) + g(X, Y)(\delta\varphi)(Z)\},$$

$$(\delta\varphi)(X) = \frac{n-p+2}{2}X(\mathrm{tr}_{12}\,\varphi)$$

for any $X, Y, Z \in C^\infty TM$. This implies that $\varphi' = \varphi - p^{-1}(\mathrm{tr}_{12}\,\varphi)g$ satisfies the equation $\delta^*\varphi' = 0$.

Finally, note that the relation $\varphi'(\dot\gamma, \dot\gamma) = \mathrm{const}$, where γ is an arbitrary geodesic in (M, g) with an affine parameter t and $\dot\gamma = \mathrm{d}\gamma/\mathrm{d}t$ is tangent to γ, is a first quadratic integral of differential equations of geodesic lines (see [6]). Thus we have proved the following.

Theorem 3.1. *If on an n-dimensional pseudo-Riemannian manifold (M, g) there exists a conformal Killing p-form and, in particular, a closed conformal Killing p-form $(1 \le p \le n-1)$, then the differential equations of geodesics have first quadratic integral.*

Choose a local orientation of M and let η be the volume element of (M, g) defined by g and the local orientation of M. Then we can define Hodge's linear operator $* : \Lambda^p M \to \Lambda^{n-p} M$ by $\omega \wedge (*\omega') = g(\omega, \omega')\eta$ for arbitrary p-forms $\omega, \omega' \in C^\infty \Lambda^p M$. We can easily verify that $\eta = \sqrt{|\det(g_{ij})|}\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ in a neighborhood U with local coordinates $\{x^1, \ldots x^n\}$.

Let us prove that the followings.

Theorem 3.2. *let (M,g) be an n-dimensional* ($n \geq 2$) *pseudo-Riemannian manifold and* $*$ *be the Hodge operator. Then we have the following isomorphism:*

$$* : \mathbf{P}^p(M, \mathbf{R}) \to \mathbf{K}^{n-p}(M, \mathbf{R}).$$

Proof. Let (M, g) be an n-dimensional ($n \geq 2$) pseudo-Riemannian manifold. Choose a local orientation of M and denote by $*$ the Hodge linear operator.

On the one hand, we assume that ω is a closed conformal Killing p-form. Then we have the following:

$$(\nabla_X(*\omega))(X_{p+1}, \dots, X_m) = \frac{1}{(p-1)!(m-p+1)} \sum_{i_2,\dots,i_p=1}^{m} \eta(X, e_{i_2}, \dots, e_{i_p}, X_{p+1}, \dots, X_m) (\mathrm{d}^*\omega)(e_{i_2}, \dots, e_{i_p}).$$

Therefore $\nabla(*\omega) \in C^\infty \Lambda^{n-p+1} M$ and, consequently, $*\omega$ is a Killing $(n-p)$-form. □

On the other hand, for a Killing p-form ω we have the following equation:

$$\nabla_j(*\omega)^{i_{p+1}\dots i_n} = \eta^{i_l\dots i_p \ i_{p+1}\dots i_n}\nabla_{[j}\omega_{i_l\dots i_p]},$$

where $1 \leq i_l < \dots < i_p \leq m$; $1 \leq i_{p+1} < \dots < i_n \leq n$; $j \neq i_l, \dots, i_p$ and also $i_a \neq i_\alpha$ for $a = 1, \dots, p$ and $\alpha = p+1, \dots, n$. Therefore it follows that $\nabla_j(*\omega)^{i_{p+1}\dots i_n} = 0$ for $j \neq i_{p+1}, \dots, i_n$ and

$$\nabla_j(*\omega)^{j i_2\dots i_n} = \eta^{i1\dots i_p j i_{p+2}\dots i_n}\nabla_{[j}\omega_{i_1\dots i_p]},$$

where the summation is not taken over j. And also we shall have

$$\eta^{l_1\dots l_p k i_{p+2}\dots i_n}\nabla_{[k}\omega_{l_1\dots l_p]} = \eta^{i_1\dots i_p j i_{p+2}\dots i_n}\nabla_{[j}\omega_{i_1\dots i_p]},$$

for the same values of the indexes $i_{p+2}, \dots, i_m$ and for $i_{p+1} = k$. Summing up the above discussion, we have

$$\nabla_j(*\omega)^{i_{p+1}i_{p+2}\dots i_n} = \frac{1}{p+1}\left\{\left(\delta_j^{i_{p+1}}\eta^{i_1\dots i_p l i_{p+2}\dots i_n} - \delta_j^{i_{p+2}}\eta^{i_1\dots i_p l i_{p+1} i_{p+3}\dots i_n} - \dots - \delta_j^{i_n}\eta^{i_l\dots i_p l i_{p+2}\dots i_{n-1} i_{p+1}}\right)\nabla_{[1}\omega_{i_1\dots i_p]}\right\}.$$

We can deduce from this equation that $*\omega$ is a closed conformal Killing $(n-p)$-form.

Remark. *For the case of* $n = 4$ *and* $p = 3$ *from Theorem* 3.1 *we obtain the known result* $* : \mathbf{K}^3(M, \mathbf{R}) \to \mathbf{P}^1(M, \mathbf{R})$ *(see* [8] *).*

In the following we shall assume the manifold (M, g) under consideration is a manifold of constant sectional curvature. Let us suppose that R^l_{ijk} are the components of the curvature tensor R of (M, g) in terms of local coordinates $\{x^1, \dots, x^n\}$. Then (M, g) satisfies the following condition (see [3]):

$$R^l_{ijk} = C(\delta^l_j g_{ik} - \delta^l_k g_{ij}), \tag{3.1}$$

where C is a constant given by $C = \mathrm{Sk}/n(n-1)$ and Sk denotes the scalar curvature of (M, g).

Now, if we use Theorem 3.2, we arrive at the following result.

Corollary 3.1. *Let (M,g) be an n-dimensional* ($n \geq 2$) *Riemannian manifold of constant nonzero sectional curvature. Then we have the following isomorphism:*

$$* : \mathbf{T}^{p}(M, \mathbf{R} \to \mathbf{T}^{n-p}(M, \mathbf{R}).$$

Proof. In an n-dimensional ($n \geq 2$) Riemannian manifold of constant sectional curvature with $\mathrm{Sk} \neq 0$, a conformal Killing p-form ω in uniquely decomposed in the form: $\omega = \omega' + \omega''$ where ω' is a Killing p-form and ω'' is a closed conformal Killing p-form (see [7,16]). Choose a local orientation of M and denote by $*$ the Hodge operator. Then by Theorem 3.2 we have $*\omega = *\omega' + *\omega''$ where $*\omega'$ is a closed conformal Killing $(n-p)$-form and $*\omega''$ is a Killing $(n-p)$-form. Therefore $*\omega$ is a conformal Killing $(n-p)$-form. The corollary is proved. □

Corollary 3.2. *Let (M,g) be an n-dimensional pseudo-Riemannian manifold of constant sectional curvature. Then* $\dim \mathbf{P}^{p}(M, \mathbf{R}) \geq n!/p!(n-p!)$.

Proof. If M is smooth manifold with a projectively flat affine connection ∇, then there are $n!/p!(n-p)!$ independent Killing p-forms (see [15]). Let us recall that a projectively flat pseudo-Riemannian manifold (M, g) is a manifold of constant sectional curvature (see [6]). Therefore, from Theorem 3.2 we can deduce that

$$\dim \mathbf{P}^{p}(M, \mathbf{R}) \geq \frac{n!}{p!(n-p)!}.$$

As a result we obtain the corollary. □

Theorem 3.3. *On an n-dimensional pseudo-Riemannian manifold (M, g) of constant nonzero sectional curvature, a closed conformal killing p-form* ω ($0 < p < n$) *is equal to the exterior differential* $\mathrm{d}\theta'$ *of a Killing* $(p-1)$*-form* θ'.

Proof. Let ω be a a closed conformal Killing p-form. Then as it is well known we have

$$\nabla_k \omega_{i_1 \ldots ip} = g_{k[i_1} \theta_{i_2 \ldots i_p]}, \tag{3.2}$$

where $\theta_{i_2 \ldots i_p} = -p(m-p+1)^{-1}(\mathrm{d}^* \omega)_{i_2 \ldots i_p}$.

Then Ricci's identity for any p-form $\omega_{i_1 \ldots i_p}$ is given by

$$\nabla_j \nabla_k \omega_{i_1 \ldots i_p} - \nabla_k \nabla_j \omega_{i_1 \ldots i_p} = -\sum_{a=1}^{p} \omega_{i_1 \ldots i_{a-1} l i_{a+1} \ldots i_p} R^{l}_{i_a jk}.$$

By virtue of Ricci identity and (3.1), we can get

$$\begin{aligned} & g_{j[i_1} \nabla_{|k|} \theta_{i_2 \ldots i_p]} - g_{k[i_1} \nabla_{|j|} \theta_{i_2 \ldots i_p]} \\ & \quad = 2C\{g_{i_1[k} \omega_{j]i_2 \ldots i_p} + g_{i_2[k} \omega_{|i_1|j]i_3 \ldots i_p} + \cdots + g_{i_p[k} \omega_{|i_1 \ldots i_{p-1}|j]}\}. \end{aligned} \tag{3.3}$$

Transvecting Eq. (3.3) with $g^{i_1 j}$, we have

$$\nabla_k \theta_{i_2 \dots i_p} = -pC\omega_{k i_2 \dots i_p}. \tag{3.4}$$

We can deduce from this equation that $\nabla\theta \in C^\infty \Lambda^p M$ and hence θ is a Killing $(p-1)$-form, i.e. $\mathrm{d}\theta = p\nabla\theta$. □

Let us assume that $C \neq 0$. If we put $\theta' = (p^2 C)^{-1}\theta$, then by virtue of Eq. (3.4) it follows that $\omega = \mathrm{d}\theta'$, which means θ' is a Killing $(p-1)$-form, the theorem is proved.

Consider a flat pseudo-Riemannian manifold (M, g) whose metric tensor is given by $g_{ij} = e_i \delta_{ij}$ with respect to local rectangular coordinates $\{x^1, \dots, x^n\}$, where $e_i = \pm 1$ and δ_{ij} are the covariant components of the Kronecker delta. The coefficients of the Killing $(p-1)$-form θ in Eq. (3.4) must now satisfy $\partial_k \theta_{i_2 \dots i_p} = 0$, where $\partial_k = \partial/\partial x^k$. Therefore θ has constant components $A_{i_2 \dots i_p}$. In this case the integrals of Eq. (3.1) take the form

$$\omega_{i_1 \dots i_p} = x_{[i_1} A_{i_2 \dots i_p]} + B_{i_1 \dots i_p}, \tag{3.5}$$

where the $x_1, \dots, x_n$ are rectangular coordinates and $B_{i_1 \dots i_p}$ are the completely skew-symmetric constant components of some p-form.

Therefore, on the basis of Theorem 3.1 we conclude, that

$$\omega_{i'_{p+1} \dots i_n} = (*\omega)_{i_{p+1} \dots i_n} = x^l A'_{l i_{p+1} \dots i_n}$$

will be the local components of a Killing $(m-p)$-form ω'. Hence we have the following.

Theorem 3.4. *Let $x^1, \dots x^n$ be the rectangular coordinates of a flat pseudo-Rienmannian manifold with the fundamental form $e_1(\mathrm{d}x^1)^2 + \dots + e_n(\mathrm{d}x^n)^2$, where $e_i = \pm 1$. Then an arbitrary p-form ω' and closed conformal Killing p-form ω on a flat pseudo-Riemannian manifold have the following components*:

$$\omega'_{i_1 \dots i_p} = x^k A'_{k i_1 \dots i_p} + B'_{i_1 \dots i_p}, \qquad \omega_{i_1 \dots i_p} = x_{[i_1} A_{i_2 \dots i_p]} + B_{i_1 \dots i_p},$$

where $A_{i_2 \dots i_p}$, $A'_{k i_1 \dots i_p}$, $B_{i_1 \dots i_p}$ and $B'_{i_1 \dots i_p}$ are components of skew-symmetric constant tensors.

Remark. *The expression, similar to the one from the theorem, for a Killing 2-form in Euclidean space E^m was deduced in the work* [17].

Let (M, g) be an n-dimensional pseudo-Riemannian manifold of class C^∞ and M' and n'-dimensional differentiable manifold of class C^∞ imbedded in (M, g) with imbedding map $f : M' \to M$. We call the image $f(M')$ a submanifold in (M, g) and identify it with the manifold M'.

Consider a neighborhood U with local coordinate system $\{x^1, \dots, x^n\}$ on M and a neighborhood U' with local coordinate system $\{u^1, \dots, u^{n'}\}$ on M' such that $f(U') \subset U$. In the local coordinates, the imbedding map f is given by $x^i = x^i(u^1, \dots, u^{n'})$, where $a, b, c, \dots, = 1, \dots, n'$.

The differential $\mathrm{d}f$ of the imbedding map $f : M' \to M$ will denoted by f_*, so that a vector field X' in M' there corresponds a vector field $f_* X'$ in M. Thus, if X' has local expression $X' = X'^a \partial/\partial u^a$, then $f_* X'$ has the local expression $f_* X' = f_a^i X'^a \partial/\partial x^i$, where $f_a^i = \partial x^i/\partial u^a$.

If we put $g' = g(f_*, f_*)$, then g' is the metric tensor $g^* = f^* g$ induced in M' from g by f, where f^* is the mapping conjugate of f_*.

The submanifold M' is called *nondegenerate* (see [3]) if the metric tensor g' is nondegenerate. Then the submanifold M' is also a pseudo-Riemannian manifold with the metric tensor g'. Hence for every point $x \in M'$ a normal subspace $(T_x M')^{\perp}$ of a tangent space $(T_x M')$, is given by the formula

$$(T_x M')^{\perp} = \{x \in T_x M : g(X, Y') = 0 \text{ for any } Y' \in T_x M'\}.$$

We have the following characteristic of this space: $(T_x M')^{\perp} \cap T_x M' = \{0\}$. Hence the orthogonal projections $h_x : T_x M \to T_x M'$ and $v_x : T_x M \to (T_x M')^{\perp}$ are defined.

We denote by ∇' the covariant differential operators corresponding to the pseudo-Riemannian metric g'. Therefore we have

$$(\nabla'_{X'} f_*) Y' = \nabla_{f_* X'} f_* Y' - f_*(\nabla'_{X'} Y') = Q(X', Y') \tag{3.6}$$

for any $X', Y' \in C^{\infty} TM'$ where a symmetric bilinear form $Q : TM' \otimes TM' \to (TM')^{\perp}$ is called *the second fundamental form*of the submanifold M' (see [6]). Eqs. (3.6) are called the equations of Gauss (see [21]).

A submanifold M' is said to be *totally umbilical* if its second fundamental form satisfies $Q = G' \otimes \mathcal{N}$ for some $\mathcal{N} \in C^{\infty}(TM')^{\perp}$.

Consider a p-form ω in a pseudo-Riemannian manifold (M, g) for $1 \le p \le n'$. We call the *projection of the p-form* ω on the submanifold M' the p-form $f^*\omega$ such that

$$(f^*\omega)(X'_1, \dots, X'_p) = \omega(f_* X'_1, \dots, f_* X'_p) \tag{3.7}$$

for an arbitrary $X'_1, \dots, X'_p \in C^{\infty} TM'$.

Now we shall prove the following.

Theorem 3.5. *For an arbitrary closed conformal Killing p-form on an n-dimensional pseudo-Riemannian manifold M its projection on an n'-dimensional* $(0 \le p \le n' < n)$ *nondegenerate umbilical submanifold M' will also be a closed conformal Killing p-form.*

Proof. From (3.7) by covariant differentiation along the submanifold M' we find that

$$\begin{aligned}
&(\nabla'_{X'} f^*\omega)(X'_1, X'_2, \dots, X'_p) \\
&\quad = (\nabla_{f_* X'}\omega)(f_* X'_1, f_* X'_2, \dots, f_* X'_p) \\
&\qquad + \omega(Q(X', X'_1), f_* X'_2, \dots, f_* X'_p) + \omega(f_* X'_1, Q(X', X'_2), \dots, f_* X'_p) \\
&\qquad + \dots + \omega(f_* X'_1, f_* X'_2, \dots, Q(X', X'_p)),
\end{aligned} \tag{3.8}$$

for an arbitrary $X'_1, \dots, X'_p \in C^{\infty} TM'$.

If we suppose that M' is totally umbilic and ω is a closed conformal Killing p-form, then (3.8) reduces to

$$
\begin{aligned}
&(\nabla'_{X'} f^*\omega)(X'_1, X'_2, \dots, X'_p)\\
&\quad = g'(X', X'_1)(f^*\theta + \omega_{\mathcal{N}})(X'_2, \dots, X'_p)\\
&\qquad - g'(X', X'_2)(f^*\theta + \omega_{\mathcal{N}})(X'_1, X'_3, \dots, X'_p)\\
&\qquad - \dots - g'(X', X'_p)(f^*\theta + \omega_{\mathcal{N}})(X'_2, \dots, X'_{p-1}, X'_1)
\end{aligned}
$$

where $\omega_{\mathcal{N}}(X'_2, \dots, X'_p) = \omega(\mathcal{N}, X'_2, \dots, X'_p)$ for an arbitrary $X'_2, \dots, X'_p \in C^\infty TM'$. This equation shows that $f^*\omega$ is a closed conformal Killing p-form on M'. □

4. Special Maxwell equations

Remember that the space–time is a smooth four-dimensional manifold M endowed with a metric tensor g of the Lorentz signature (1,3).

Consider the electromagnetic field tensor F of the space–time (M, g). Let U be a neighborhood with local coordinate system $\{x^0, x^1, x^2, x^3\}$ on (M, g). In the local coordinates, the electromagnetic field tensor F is given by $F_{ij} = F(\partial_i, \partial_j)$, where $\partial_i = \partial/\partial x^i$ and $i, j, k, \dots = 0, 1, 2, 3$.

As it is well known, the 2-form $\omega = 2^{-1} F_{ij}\, \mathrm{d}x^i \wedge \mathrm{d}x^j$ of the electromagnetic field is closed, which is equivlanet to the conditions $\omega \in \ker \mathrm{d}$.

Besides, we consider two more incompatible conditions on the 2-form ω. On the one hand, the electromagnetic form ω is assumed to be divergence-free, which corresponds to the Maxwell equations in a free space: $\mathrm{d}^*\omega = 0$. On the other hand, it is required that $\mathrm{d}^*\omega \neq 0$. In this case, the Maxwell equations $(\mathrm{d}^*\omega)(X) = 4\pi g(X, \mathcal{J})$ for any $X \in C^\infty TM$ and the current 4-vector $\mathcal{J}$ in the presence of a charge are associated.

If we apply the representation theory of orthogonal group here, then the existence of these incompatible cases becomes obvious from the mathermatical point of view as well.

Let $C^\infty \Lambda^2 M$ bed a space of all C^∞-sections of $\Lambda^2 M$ over four-dimensional Lorentz manifold (M, g). According to Theorem 2.1 there exist only three fundamental differential operators of order one determined on $C^\infty \Lambda^2 M$. They are $\mathrm{D}_1 = 3^{-1} d$; $\mathrm{D}_2 = 3^{-1} g \Box \mathrm{d}^*$ and $\mathrm{D}_3 = \nabla - 3^{-1}(D + g\Box \mathrm{d}^*)$.

In particular, if an exterior differential 2-form belongs to $\ker \mathrm{D}_1$, it is said to be closed. We have already stated the space of a closed 2-forms by $\mathbf{D}^2(M, \mathbf{R})$.

We have two important subspaces of $\mathbf{D}^2(M, \mathbf{R})$ in a Lorentz manifold (M, g). These subspaces will be $\mathbf{H}^2(M, \mathbf{R})$, consisting of closed coclosed 2-forms on (M, g), and $\mathbf{P}^2(M, \mathbf{R})$, consisting of closed conformal Killing 2-forms on (M, g). In other words, $\omega \in \mathbf{H}^2(M, \mathbf{R})$, if and only if $\omega \in \ker \mathrm{D}_1 \cap \ker \mathrm{D}_2$, and $\omega \in \mathbf{P}^2(M, \mathbf{R})$, if and only if $\omega \in \ker \mathrm{D}_1 \cap \ker \mathrm{D}_3$.

On the one hand, in the case where the 2-form ω of the electromagnetic field belongs to $\mathbf{H}^2(M, \mathbf{R})$, is associated with the Maxwell equations in a charge-free space ($\mathrm{d}^*\omega = 0$).

On the other hand, if the 2-form ω of the electromagnetic field belongs to $\mathbf{P}^2(M, \mathbf{R})$ we have the following equations:

$$\nabla\omega = \tfrac{1}{3} g \Box \mathrm{d}^*\omega, \tag{4.1}$$

which we call the *special Maxwell equations*.

Let's define the vector field $\mathcal{J} \in C^{\infty}TM$, suth that

$$(\mathrm{d}^*\omega)X = -4\pi g(\mathcal{J}, X). \tag{4.2}$$

Then from Eqs. (4.1) we obtain:

$$(\nabla_X\omega)(Y, Z) = \frac{4\pi}{3}[g(X, Z)g(\mathcal{J}, Y) - g(X, Y)g(\mathcal{J}, Z)]. \tag{4.3}$$

Let's remark, that equations. (4.3) automatically imply the Maxwell equations in the presence of a charge.

Theorem 4.1. *Let the 2-form ω of the electromagnetic field be a conformal Killing form. Then*

1. *the tensor F of the electromagnetic field will be parallel along integral curves of the current 4-vector $\mathcal{J}$;*
2. *the 3-form of charge $*\mathcal{J} = 3\nabla(*\omega)$ for the 2-form ω of the electromagnetic field;*
3. *the current 4-vector $\mathcal{J}$ will be a Killing vector, if the space–time (M, g) is an Einstein manifold;*
4. *the energy–momentum tensor T for the electromagnetic field is a conformal Killing tensor.*

Proof. At first, the Maxwell equations (4.3) can be represented locally by

$$\nabla_i F_{jk} = \frac{4\pi}{3}(\mathcal{J}_j g_{ik} - \mathcal{J}_k g_{ij}) \tag{4.4}$$

from which $\mathcal{J}^k \nabla_k F_{ij} = 0$. Hence the electromagnetic field tensor F is parallel along integral curves of the current 4-vector $\mathcal{J}$.

Secondly, in electromagnetism (see [9]) the exterior differential of the 2-form $*\omega$ gives 3-form of the charge $*\mathcal{J}$, multiplied on 4π. As the 2-form ω is a closed conformal Killing form, then, according to the Theorem 1.1 from the first paragraph, the 2-form $*\omega$ will be a Killing form. Therefore it can be written as $*\mathcal{J} = 3\nabla(*\omega)$.

Thirdly, we use the following Ricci identites for the tensor F in coordinate form (see [6]):

$$\nabla_i\nabla_j F_{kl} - \nabla_j\nabla_i F_{kl} = -F_{ml}R^m_{kij} - F_{km}R^m_{lij}. \tag{4.5}$$

Substituting (4.4) for (4.5), we have

$$\frac{4\pi}{3}(g_{jl}\nabla_i\mathcal{J}_k - g_{jk}\nabla_i\mathcal{J}_l - g_{il}\nabla_j\mathcal{J}_k + g_{ki}\nabla_j\mathcal{J}_l) = -F_{ml}R^m_{kij} - F_{km}R^m_{lij}. \tag{4.6}$$

By transvecting g^{jl} to (4.6), it follows that

$$\nabla_i\mathcal{J}_k = \frac{3}{8\pi}(F^{ml}R_{mlki} - F_{km}R^m_i), \tag{4.7}$$

from which we have $\nabla_j\mathcal{J}^i = 0$.

Since (M, g) is an Einstein manifold, i.e. Ric $= 4^{-1}$ Sk g, we get from (4.3) that $\mathcal{J}$ is a Killing vector field.

And fourthly, it is easy to verify that the symmetric 2-form φ with the components

$$\varphi_{ij} = \frac{1}{4\pi}\left[g^{kl}F_{ik}F_{jl} - \tfrac{1}{2}(F^{kl}F_{kl})g_{ij}\right]$$

satisfies the equation $\delta^*\varphi = 0$ by means of Theorem 3.1. Therefore, φ is a Killing symmetric 2-form. Furthermore, direct calculations show that the symmetric 2-form φ' with the components of the energy–momentum tensor of the electromagnetic field (see [9]).

$$T_{ij} = \frac{1}{4\pi}\left[F_{ik}F_j^k - \tfrac{1}{4}(F^{kl}F_{kl})g_{ij}\right]$$

is the traceless part of the symmetric 2-form φ and, therefore, is a conformal Killing symmetric 2-form (see [8]).

Consider the flat space–time with a Lorentz coordinate system $\{x^0, x^1, x^2, x^3\}$. It is clear the Eq. (4.7) will be rewritten as

$$\frac{\partial \mathcal{J}_j}{\partial x_i} = 0$$

for all $i, j = 0, 1, 2, 3$. Then the integrals of Eq. (4.4) will take the form:

$$F_{ij} = \frac{4\pi}{3}(C_i x_j - C_j x_i + C_{ij}) \tag{4.8}$$

for $\mathcal{J} = \{C_0, C_1, C_2, C_3\}$ and arbitrary constants C_i and $C_{ij} = -C_{ji}$. Thus, we have proved the followed theorem. □

Theorem 4.2. *The integrals of the special Maxwell equations in a flat space–time with a Lorentz coordinate system* $\{x^0, x^1, x^2, x^3\}$ *have the form*

$$F_{ij} = \frac{4\pi}{3}(C_i x_j - C_j x_i + C_{ij}),$$

where C_i are components of the current 4-vector $\mathcal{J}$ and C_{ij} are arbitrary components of skew-symmetric 2-tensor C.

Using the classical notations (see [9]), we introduce the electric and magnetic field intensity vectors **E** and **B**, by setting $\mathbf{E} = \{F_{10}, F_{20}, F_{30}\}$ and $\mathbf{B} = \{F_{23}, F_{31}, F_{12}\}$. If in addition, x^0 is interpreted as time t, then we have

$$\frac{\partial \mathbf{E}}{\partial t} = \{-C_1, -C_2, -C_3\} \quad \text{and} \quad \frac{\partial \mathbf{B}}{\partial t} = \{0, 0, 0\}.$$

According to Theorem 3.5, the projection $f^*\omega$ of a closed conformal Killing form ω on a nondegenerate umbilical submanifold is a flat form. Hence we have the following theorem.

Theorem 4.3. *The projection $f^*\omega$ of the electromagnetic field 2-form ω on any 3-dimensional space-like totally umbilical submanifold obeys the special Maxwell equations, if the 2-form ω obeys these equations.*

We consider a four-dimensional Lorentz manifold (M, g) with a global defined vector field ξ such that $g(\xi, \xi) = -1$. In this case the Lorentz manifold (M, g) is called *time oriented space–time* (see [3]).

Direct verification shows that every tangent space T_xM is the orthogonal sum of the *horizontal subspace* $\mathcal{H} = \{X \in T_xM, g(X, \xi_x) = 0\}$ and the *vertical subspace* $\mathcal{V} = \text{span}\,\{\xi_x\}$. Denote by $v_x : T_xM \to \mathcal{V}$ and $h_x : T_xM \to \mathcal{H}$ orthogonal projections such that $v(X) = -g(X, \xi)$ and $h(X) = X + g(X, \xi)$ for any $X \in C^\infty TM$. Then $q = q^v \oplus q^h$ and $O(q) = Q(q^v) \times Q(q^h)$ for $q^v = q(v, v)$ and $q^h = q(h, h)$, where q^h is a positive define quadratic form on the three-dimensional subspace $\mathcal{H}$. Then we have the orthogonal sum

$$\omega = \omega^h \oplus \omega^v \tag{4.9}$$

for the electromagnetic field 2-form ω, the magnetic field intensity 2-form $\omega^h = \omega(h, h)$ and the electric field intensity 2-form

$$\omega^v = \frac{m}{\rho}(g(X, \xi)g(Y, \xi') - g(Y, \xi)g(X, \xi')),$$

where m is the particle mass, ρ the charge, ξ the 4-velocity vector and ξ' the 4-acceleration vector (see [11,14]). This sum (4.9) is irreducible under the induced representation of $O(q) = Q(q^v) \times O(q^h)$ on $\Lambda^2 T_xM$.

Consider the distribution $\xi^\perp$ on (M, g) as a smooth field of three-dimensional spacelike tangent subspaces, that is, a function whose value at each point $x \in M$ is a three-dimensional spacelike subspace $\mathcal{H}$ of the tangent space T_xM. Let the distribution $\xi^\perp$ be integrable; then through each point $x \in M$ there exists a three-dimensional integral submanifold M' which is tangent to the distribution at each of its points. Each integral submanifold M' is a Riemannian manifold (M', g') where $g' = f^*g$ is a positive definite metric. In this case we can see that the space–time (M, g) admits a 3 +1 split.

Furthermore, we suppose that the submanifold (M', g') is totally umbilical. Recently, the Gauss equations suggested us to introduce the condition.

$$g(Q(X', Y'), Q(Z', W')) = g(H, H)g'(X', Y')g'(Z', W') \tag{4.10}$$

for the timelike mean curvature vector field $H = 3^{-1}$ trace Q of (M', g') and any vector fields X', Y', Z', W' of $C^\infty TM'$. We may state that, using (4.10), it is immediate from Gauss equations to derive relations

$$\mathcal{K}'_\pi = \mathcal{K}_\pi + g(H, H) \tag{4.11}$$

linking the sectional curvature $\mathcal{K}'_\pi$ of (M', g') with the corresponding sectional curvature $\mathcal{K}_\pi$ of (M, g) in direction of any 2-plane $\pi \subset T_xM'$ (see [3]).

We note that the projection $f^*\omega$ of the closed conformal Killing 2-form ω of the electromagnetic field on any totally umbilical integral submanifold of the distribution $\xi^\perp$ is the closed conformal Killing 2-form ω^h of the magnetic field intensity. Thus it is known that a compact m-dimensional Riemannian manifold (M', g') of nonpositive sectional curvature $\mathcal{K}'_\pi$ does not admit closed conformal Killing p-forms $(1 < p < m)$ with nonconstant norm (see [13]). Hence we have by virtue of Eq. (4.11), the following theorem.

Theorem 4.4. *If a time-oriented space–time (M, g) of nonpositive spacelike sectional curvature admits a 3 +1 split with compact spacelike totally umbilical hypersurfaces and the* *electromagnetic field 2-form ω is a conformal Killing form then the norm of the magnetic field intensity form ω^h is constant along an arbitrary hypersurface.*

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